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EVALUATION OF STEADY BASIC FLOW IN A SPHERICAL COUETTE SYSTEM USING A PERTURBATION METHOD

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Abstract
In this paper we present the behavior of a viscous, incompressible Newtonian flow between concentric spheres when one of the which is held fixed and the other is rotating at an arbitrary angular velocity, and also when both spheres are rotating. The two control parameters that govern this phenomenon are, the dimensionless gap width between the spheres, and the Reynolds-number. An analytic solution was obtained using a theoretic "regular perturbation" method by means of which we analyze particular cases. The results were obtained through the projection of the meridional streamlines, contours of constant angular azimuthal velocity, and the evolution of secondary flow. This allowed us to show that the steady basic flow is axisymmetric and reflection-symmetric. This kind of flow could be of interest for understanding global astrophysical and geophysical processes that take into account the rotation and spherical geometry.
Keywords: rotation; concentric spheres; basic flow; perturbation method; gap width; Reynolds-number.

1. Introduction
The physical phenomenon that implicate the flow between two concentric rotating spheres is governed for parameters as Reynolds-number, the clearance-ratio between the spheres, angular velocities of the spheres, density and viscosity of the fluid [1-5].
In this paper it is analyzed two cases: a) when both spheres are rotating [5], and when the outer sphere is held fixed and the inner one is rotating [4]; both cases at an arbitrary angular velocity, and considering steady-state spherical flow in "wide gap" and "small gap" geometries, with the purpose of analytically simulate oscillating primary and secondary flows [1-2]. All this, on the basis of the Navier-Stokes equations [6], which allow us obtain a streamlines function for determine the components of the secondary flow field. Of the primary flow, we show contours of constant angular azimuthal velocity [1], and of the secondary flow, the projection of the streamlines and the evolution of the components of the secondary flow in both directions, radial and meridional. Neither Taylor-Görter (TG) vortices occur nor laminar-turbulent transition in our spherical Couette system as reported by [3].
Although the spherical Couette flow is relevant to most astrophysical, geophysical, and engineering applications, it has been studied less because it is more difficult to treat analytically [1], however we represent this flow through an analytic "regular perturbation" method that takes into account two terms. Of the results obtained, we could observe that, in the wet surface of the both spheres exist stagnation points where both components of the secondary flow are zero. The rigid rotation case looks like a singular limit when both spheres have the same angular velocity. Sawatzki & Zierep [7], and Khlebutin [8], found experimentally that at low Reynolds numbers spherical Couette flows are both axisymmetric and reflection-symmetric about the equator, like effectively we found with our theoretician "regular perturbation" method. There are axisymmetric higher flow modes in spherical Couette flow [1-3] which are out of reach of the present study.

2. Method of solution

In order to evaluate the flow of an incompressible viscous fluid of density ρ and viscosity μ at constant temperature which is contained between two concentric spheres rotating about a common axis, we evaluate the primary (Stokes solution) and secondary flows regimes to the radii kR and R and angular velocities W_i and W_0 of the inner and outer spheres, respectively (see Fig. 1). The Stokes solution is the Stokes limit:

$$v_\phi=\frac{kR\sin\theta}{1-k^3}\left[\frac{ar}{kR}+b\left(\frac{kR}{r}\right)^2\right] \tag{1}$$

Where $a=W_0-k^3W_i$ and $b=W_i-W_0$. Equation (1) was obtained through the boundary conditions (see Fig. 1): $v_r=0$, $v_\theta=0$, $v_\phi=R\,W_0\sin\theta$ at $r=R$ and $v_r\to 0$, $v_\theta\to 0$, $v_\phi=kR\,W_i\sin\theta$ at $r=kR$[1]. We evaluate the secondary flow using "regular perturbation" method, that takes into account the perturbation parameter that describes the importance of the inertial effects, that is, the Reynolds-number. For low Reynolds-number, the perturbative terms are;

$$\boldsymbol{v}=v_0+Re\,v_1+\ldots;\quad p=p_0+Re\,p_1+\ldots \tag{2}$$

where $\boldsymbol{v}$ is the velocity field, p the pressure field, $\boldsymbol{v_0}\,(r,\theta)$ is the velocity field given by Eq. (1), and $\boldsymbol{v_1}$ must be zero on the wetted surfaces of the spheres. Here, the Navier-Stokes equation for the ϕ–component in spherical coordinates becomes;

$$-\frac{\rho(R-kR)^2(W_0-W_i)}{\mu}\frac{\mu}{r^2}\left[\frac{\partial}{\partial r}r^2\left(\frac{\partial}{\partial r}v_\phi\right)+\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[v_\phi\sin\theta\right]\right)\right]+\frac{\rho(R-kR)^2(W_0-W_i)}{\mu}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}p+\frac{\mu}{b(R-kR)^2}\left[v_\phi\frac{\partial}{\partial\phi}v_\phi\right]=0 \tag{3}$$

Where the fourth term of the left represents small inertial effects and will be called $\boldsymbol{f}$. Then;

$$\vec{f}=\frac{\mu k^2R^2}{b(R-kR)^2(1-k^3)^2}\left[\frac{a^2r}{k^2R^2}+\frac{2abkR}{r^2}+\frac{b^2(kR)^4}{r^5}\right]\sin^2\theta\,\vec{\delta}_r+\sin\theta\cos\theta\,\vec{\delta}_\theta \tag{4}$$

Where $\boldsymbol{\delta_r}$ and $\boldsymbol{\delta_\theta}$ are the unit vectors on the r and θ directions[2]. f and $\boldsymbol{v_1}$ have the same properties. We derive $\boldsymbol{v_1}$ from a stream function $\psi(r,\theta)$ for axisymmetrical spherical flow. The form of ψ is;

$$\psi=f(r)\sin^2\theta\cos\theta \tag{5}$$

The general solution of $f(r)$ is;

$$f(r)=-\frac{1}{14700}\frac{f_1(r)}{f_2(r)}+C_i \tag{6}$$

where:

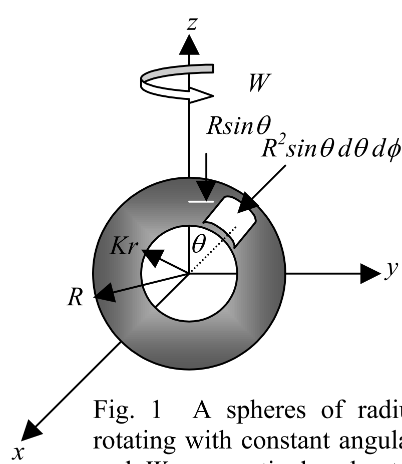

Fig. 1 A spheres of radius kR and R rotating with constant angular velocity W_i and W_0, respectively, about the z-axis in an incompressible Newtonian fluid.

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[1] The only terms of the Navier-Stokes equation that survive are:

$$0=\frac{\mu}{r^2}\left[\frac{\partial}{\partial r}r^2\left(\frac{\partial}{\partial r}v_\phi\right)+\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[v_\phi\sin\theta\right]\right)\right]$$

[2] Thus, $f(r)$ must satisfy the differential equation of fourth order;

$$\frac{\partial^4}{\partial r^4}f(r)-\frac{12}{r^2}\frac{\partial^2}{\partial r^2}f(r)+\frac{24}{r^3}\frac{\partial}{\partial r}f(r)=\frac{2b^2k^6R^6-2abk^3R^3r^3-4a^2r^6}{(R-kR)^2(1-k^3)^2br^5}$$

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$$f_1(r) = -708a^2r^6 + 1225abk^3R^3r^3 + 1225b^2k^6R^6 + 40a^2r^6\ln(r)$$

$$f_2(r) = bR^2r\left(1 - 2k^3 + k^6 - 2k + 4k^4 - 2k^7 + k^2 - 2k^5 + k^8\right)$$

$$C_i = C_1 + C_2r^{-2} + C_3r^3 + C_4r^5$$

We note that $f(r)$ present a singularity when the angular velocities in the spheres is the same. C_i are constants. The components of the secondary flow field, v_I, are evaluated by the expressions:

$$v_r = -\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \tag{7}$$

$$v_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \tag{8}$$

This components were used by Zikanov [5] to display the properties of the calculated axisymmetric flows in base to a meridional streamfunction.

3. Results and discussion

In Table I we can see some of the particular cases analyzed in this research. In the first case the outer sphere is static while the inner one rotates with an angular velocity of 100. The cases 2 to 4 represent the situation were both spheres rotates at almost the same velocity, differing the cases 2 and 3 of the 4 on the magnitude of these velocities. The numerical results were computed using the dimensionless gap width of 0.11 for the cases 1 and 2 and of 0.39, for the cases 3 and 4. Here only it is shown the results of case 1 and 4. For the case 1, the Fig. 2 shows streamlines, and the primary and secondary flows. In this case, both flows are comparables. In the case 3 (Fig. 3), the meridional streamlines and the component of the secondary flow in the meridian direction, a fluid particle moves along a streamline near of the inner sphere and after this, is thrown centrifugally outward; the component of the secondary flow in radial direction present reflection-symmetric and the secondary flow is large, whereas the primary flow only represents the 3.5% of those approximately. In neither case appears Taylor-Görter (TG) vortices because of the Reynolds-number used, although in the first two cases we used a "small gap" where, for high Reynolds-number, those occur [1-5].

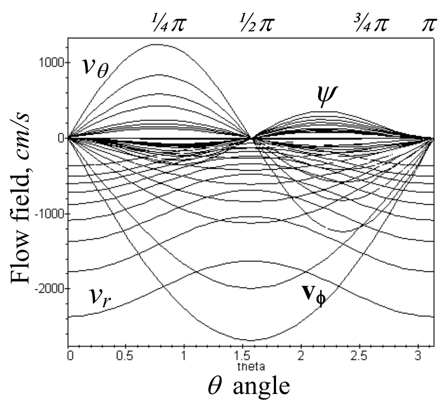

Fig. 2 Evolution of the streamlines, ψ, primary, v_ϕ, and secondary, v_θ and v_r, flows, under conditions of the case 1

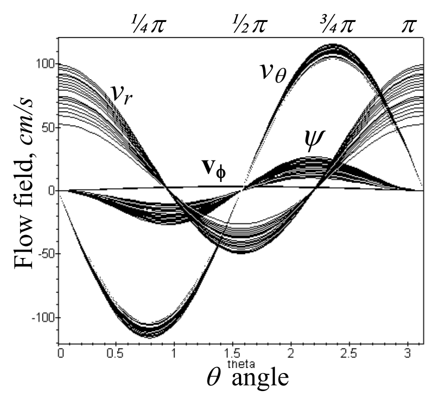

Fig. 2 Evolution of the streamlines, ψ, primary, v_ϕ, and secondary, v_θ and v_r, flows, under conditions of the case 3

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Table 1 Conditions of the cases considered.			
Cases	Gap ratio [dimensionless]	W_i [rad/s]	W_0 [rad/s]
1	0.11	100π	0
2	0.11	0.95π	π
3	0.39	0.95π	π
4	0.39	950π	1000π

4. Conclusions

The cases analyzed in this research show typical situations for steady-states axisymmetric spherical Couette flow at gap ratios of 0.11 and 0.39. All graphics show that the steady basic flow is axisymmetric and reflection-symmetric and consists of differential rotation about the axis and circulation in the meridional plane to the clearance-ratios used between the spheres, as we expected, given the prior researches of this physical phenomenon. The stream lines and the component of the secondary flow, v_θ, shows a sinusoidal behavior in all cases and the primary flow has most of its spatial structure near the equator. For the case of the thin layer, σ = 0.11, and both spheres rotating, the secondary flow is large compared with the primary one. Then, we conclude that, the innertial effects are present still to low Reynolds-number.

The analytic solution for the flow between both spheres in the Stokes limit using the perturbation method is limited to two terms in the perturbative expansion. Although we know that the Stokes' law of creeping motion establish inertia negligible, all the cases analyzed here show small inertial effects.

In spite of have chosen the gap width of 0.11 between the spheres, we never observe Taylor-Görter vortices in the flow, this because of the low Reynolds-number used. For this number, the contours of constant angular velocity are approximately parallel to the spherical boundaries as has been reported elsewhere. There are stagnation points as in the poles as the equator with respect at the meridional streamlines and the secondary flow, v_θ., Ekman pumping causes fluid to be thrown outward centrifugally along the inner sphere and pulled from the center of the outer one.

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